Proof of UVA10209

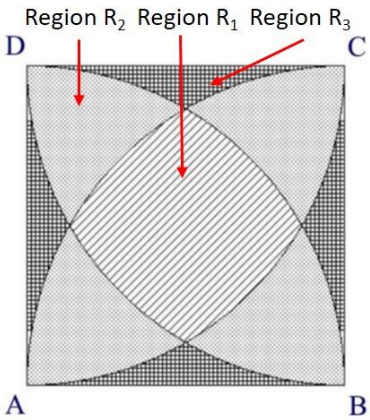

Figure 1. Three regions: R_1, R_2, and R_3

Let the region of the three shapes be R_1, R_2, and R_3 as shown in Figure 1.

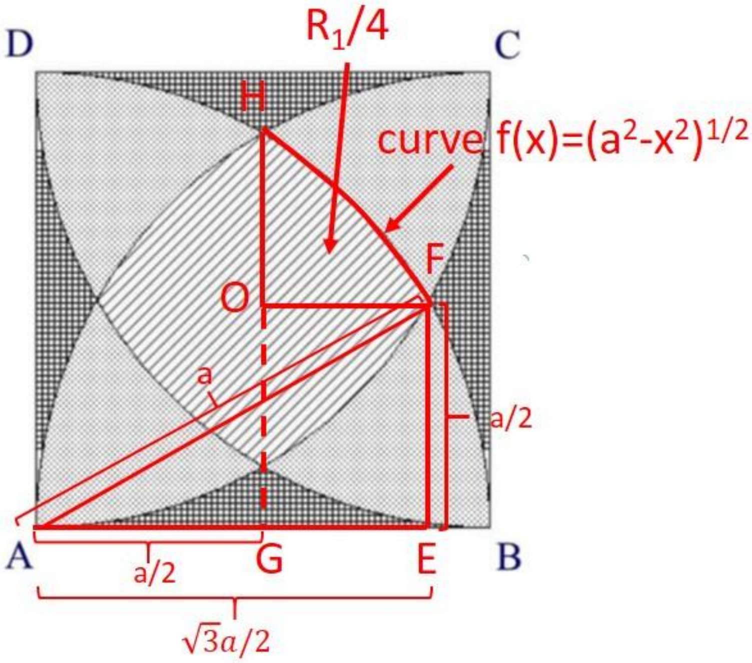

Figure 2. Area of OHF, area(R_2)/4

We first consider region R_1 as shown in Figure 2. The side of the square is a.

Consider triangle AEF, AF=a, EF=a/2. Therefore, $\mathrm{AE} = \sqrt{a^2 - (\frac{a}{2})^2} = \sqrt{3}a/2$.

The area of OHF is the integral of curve $f(x)=(a^2-x^2)^{1/2}-a/2$ from a/2 to $\sqrt{3}a/2$.

Hence the quarter area of region R_1 is

$$\mathrm{area(R1)}/4 = \int_{a/2}^{\sqrt{3}a/2} \left(\sqrt{a^2 - x^2} - a/2\right) dx$$

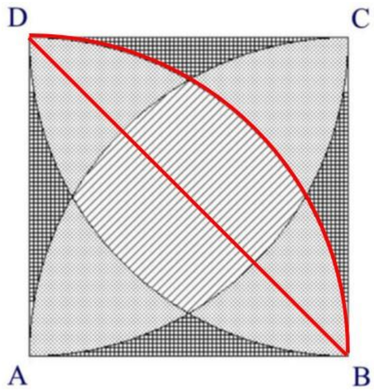

Figure 3. area(R_1)/2+area(R_2)/4

From Figure 3, we obtain the equations area(R_1)/2+area(R_2)/4=$(\pi a^2 - 2a^2)/4$. Furthermore area(R_1)+area(R_2)+area(R_3)=a^2. Hence, we can compute the areas of regions: R_1, R_2, and R_3.

Q.E.D.