105 學年 第一學期
離散數學 第二次平時考試 資訊二甲
學號： **姓名：Solutions**

1. (20 points) (a) 集合 A 的定義如下，寫出 A 的冪集合 (power set) $\mathscr{P}$(A)。(Set A is defined as below. Write the power set of A, $\mathscr{P}$(A).)

   A = {1, 2, {1, 2}}

   (b) 假設 X, Y, 和 Z 是三個 $\mathscr{P}$(A) 中的不同元素 (elements)，且 X∩(Y∪Z)={1, 2}. 寫出三組不同的 X, Y, 和 Z。(Let X, Y, and Z be three various elements in $\mathscr{P}$(A) and X∩(Y∪Z)={1, 2}. Write three different solutions of X, Y, and Z.)

   (a) $\mathscr{P}$(A)={∅, {1}, {2}, {{1, 2}}, {1, 2}, {1, {1, 2}}, {2, {1, 2}}, {1, 2, {1, 2}}}.

   (b) (i) X={1, 2}, Y={1}, Z={2}; (ii) X={1, 2}, Y=∅, Z={1, 2, {1, 2}}; (iii) X={1, 2, {1, 2}}, Y={1}, Z={1, 2}.

2. (30 points) 假設 g 是自 A 映到 B 的函數及 f 是 B 映到 C 的函數，分別考慮下列兩個敘述。如果該敘述為真，則證明之；否則，則舉出一個反例。(Let g be a function from A to B and f be a function B to C. For each of the following two statements, if it is true, prove; otherwise, give a counterexample.)

   (a) 若 f 和 g 都是映成函數，則 f∘g 是一個映成函數。(if f and g are onto functions, f∘g is an onto function.)

   (b) 若 f∘g 是一個一對一函數，則 f 和 g 都是一對一函數。(if f∘g is a one-to-one function, f and g are one-to-one functions,.)

   (a) Let c be an element in C. Since f is an onto function, there must be some element b in B such that f(b)=c. Also, since g is an onto function, there must be some element a in A, such that g(a)=b. We have f∘g(a)=f(g(a))=f(b)=c. Hence, for an element c in C, there is some a in A such that f∘g(a)=c, i.e., f∘g is an onto function from A to C.

   (b) Let A={1, 2, 3}, B={a, b, c ,d}, C={x, y, z}. If g(1)=a, g(2)=b, and g(3)=c, then g is a one-to-one function; if f(a)=x, f(b)=y, f(c)=z, and f(d)=z, then f is not a one-to-one function. However, f∘g(1)=x, f∘g(2)=y, and f∘g(3)=z such that f∘g is a one-to-one function. We have a counterexample that f∘g is a one-to-one function, but f is not.

3. (20 points) 假設 A 和 B 是兩個集合，證明 $(A \cap B) \cup (A \cap \overline{B}) = A$。(Let A and B be two sets, prove $(A \cap B) \cup (A \cap \overline{B}) = A$.)

   $A = A \cap U$ (Identity Law)

   $= A \cap (B \cup \overline{B})$ (Complement Law)

   $= (A \cap B) \cup (A \cap \overline{B})$ (Distributive Law)

4. (30 points) (a) 定義對等關係 (equivalence relation) (Define equivalence relation.)

   (b) 假設 f:A→B 是一個函數，關係 R={(x, y) | x, y∈A 且 f(x)=f(y)}；證明 R 是一個對等關係。(Let f:A→B be a function and relation R={(x, y) | x, y∈A and f(x)=f(y)}. Prove R is an equivalence relation.)

   (c) 如果 (b) 小題的 f 是一個整數到整數的函數，且 $f(x)=x^2$, 寫出關係 R 所產生的對等類別 $[a]_R$, a∈A。(Suppose f in (b) is a function from Z to Z and f(x)= $=x^2$. Write the equivalence class $[a]_R$, generated by relation R, a∈A.)

   (a) A relation R⊆A×A is an equivalence relation if it satisfies the following laws:

      i. Reflexive law: x∈A ⇒ (x, x)∈R.
      ii. Symmetric law: x, y∈A ∧ (x, y)∈R ⇒ (y, x)∈R.
      iii. Transitive law: x, y, z∈A ∧ (x, y)∈R ∧ (y, z)∈R ⇒ (x, z)∈R.

   (b) Show R is and equivalence equation.

i. Reflexive law: For an element x of A, since f(x)=f(x), we have (x, x)∈R.
ii. Symmetric law: For x, y∈A ∧ if (x, y)∈R, then we have f(x)=f(y). Hence f(y)=f(x), we obtain (y, x)∈R.
iii. Transitive law: For x, y, z∈A ∧ (x, y)∈R ∧ (y, z)∈R, i.e., f(x)=f(y) and f(y)=f(z). Thus, we have f(x)=f(z) and conclude (x, z)∈R.

(c) $[a]_R=\{x \mid x\in A \wedge |x|=a\}$, e.g., $[0]_R=\{0\}$, $[1]_R=\{-1, 1\}$, $[n]_R=\{-n, n\}$.