105 學年 第一學期
離散數學 第三次平時考試 資訊二甲、資訊二乙
學號： 　　　　　　　　**姓名：Solutions**

1. (30 points) 假設 f(x) 和 g(x) 是整數函數，定義：(Let f(x) and g(x) be integer functions, Define:)
   (a) f(x) 是 O(g(x)) [f(x) is O(g(x))],
   (b) f(x) 是 Ω(g(x)) [f(x) is Ω(g(x))],
   (c) f(x) 是 Θ(g(x)) [f(x) is Θ(g(x))].
   (a) Let f and g be integer functions. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that $|f(x)| \leq C|g(x)|$, whenever $x>k$.
   (b) Let f and g be integer functions. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that $|f(x)| \geq C|g(x)|$, whenever $x>k$.
   (c) Let f and g be integers functions. We say that $f(x)$ is $\Theta(g(x))$, if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.

2. (30 points) 假設 r 是一個正整數常數，(Let r be a positive integer constant,)
   (a) 寫出最小參考函數 g(n), 使得 $\sum_{k=1}^{n}(-1)^k k^r$是 O(g(n)), (Write the "least" reference function g(n) such that $\sum_{k=1}^{n}(-1)^k k^r$ is O(g(n)),)
   (b) 寫出一組 witness C 和 k 以證明 $\sum_{k=1}^{n}(-1)^k k^r$ 是 O(g(n)). (Write a pair of witness C and k to prove $\sum_{k=1}^{n}(-1)^k k^r$ is O(g(n)).)
   (a) The least reference function is g(n) = n^{r+1}.
   (b) If n>1, $|\sum_{k=1}^{n}(-1)^k k^r| = \sum_{k=1}^{n}|(-1)^k k^r| = \sum_{k=1}^{n} k^r \leq \sum_{k=1}^{n} n^r = n \times n^r = n^{r+1}$. Choose witness C=1 and k=1, by the definition of big-O, we have function $\sum_{k=1}^{n}(-1)^k k^r$ is $O(n^{r+1})$.

3. (40 points) 對下列每一個函數找出其 big-O 函數的最小參考函數 (For each of the following functions find the least reference function of its big-O function):
   (a) $n \log(n^2 + 1) + n^2 \log(n)$
   (b) $n^{2^n} + n^{n^2}$
   The three theorems are used for the following proofs:
   Theorem 1: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real numbers. We have $f(x)$ is $O(x^n)$.
   Theorem 2: Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. We have $(f_1+f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.
   Theorem 3: Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. We have $(f_1 f_2)(x)$ is $O(g_1(x) g_2(x))$.
   (a) By Theorem 1, we have n^2+1 is $O(n^2)$. Hence, $n \log(n^2 + 1)$ is $O(n \log(n^2))$. It is known that $n \log(n^2)=2n \log(n) \leq n^2 \log(n)$, for n>1. By Theorem 2, the big-O function of function $n \log(n^2+1)+n^2 \log(n)$ is $\max(O(n \log(n^2)), O(n^2 \log(n)))=O(n^2 \log(n))$.
   (b) If $n>4$, we know that $2^n>n^2$. Hence, $n^{2^n} > n^{n^2}$, $\forall n$: $n>4$. By Theorem 2, the big-O function of $n^{2^n} + n^{n^2}$ is $\max\left(O\left(n^{2^n}\right), O\left(n^{n^2}\right)\right) = O\left(n^{2^n}\right)$. The final solution is $O\left(n^{2^n}\right)$.