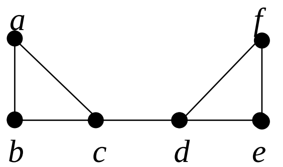
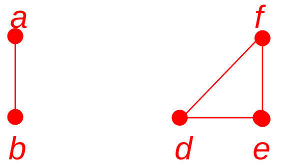
105 學年 第一學期 第四次平時考試
離散數學 資訊二甲、資訊二乙
學號： **姓名：Solutions**

1. (30 points) 假設在一個圓週上以任何順序寫下十個整數 1, 2, 3, 4, 5, 6, 7, 8, 9, 和 10，證明其中三個相鄰整數的和大於或等於 17. (Suppose write ten integers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 on the circumference of a circle in any order. Prove that there are 3 neighboring integers whose sum is greater than or equal to 17.) (提示：使用鴿籠原理，考慮所有三個相鄰整數的個數及所有整數的和。(Hint: Use Pigeon Hold Principle. Consider the number of cases for all 3 neighboring integers and the sum of all integers.))
Assume the sum of any three neighboring integers on the circle is not greater than or equal to 17. Let the ten integers on a circle in order starting from an element, say, a_0, be $a_0, a_1, a_2, \ldots, a_9$. If we construct a set of triples cyclically $\{(a_k, a_{k+1\,\%\,10}, a_{k+2\,\%\,10}) \mid 0 \le k < 10\}$, there are 10 triples. Each element a_k in the ten triples appear exactly three times in the set. Let s_k be $a_k + a_{k+1\,\%\,10} + a_{k+2\,\%\,10}$. By the assumption, we have $s_k \le 16$, for all k such that $0 \le k < 10$. Hence, $\sum_{k=0}^{9} s_k \le 16 \times 10 = 160$. However, $\sum_{k=0}^{9} s_k = 3 \times \sum_{k=1}^{10} k = 3 \times 55 = 165$. It is a contradiction by Pigeon Hole Principle. That is, it is not possible to place 165 objects in 160 boxes with no any box having more than one objects. Hence, there must exist a sum of three neighboring numbers greater than or equal to 17.

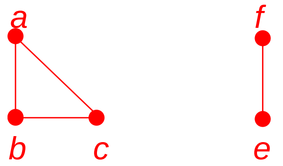
2. (30 points) (a) 定義何謂 cut vertex 和 cut edge。(Define cut vertex and cut edge.) (b) 寫出下圖所有的 cut vertices 和 cut edges. (Find all the cut vertices and cut edges of the following graph.)



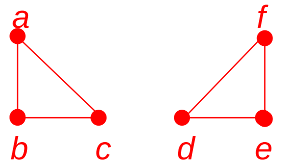
Removing either vertex *c* or vertex *d*, the graph will leave two components. Hence, either vertex *c* or vertex *d* is a cut vertex.
Removing edge {c, d}, the graph will leave two components. Hence, edge{*c*, *d*} is a cut edge.



Remove vertex *c* from the graph.

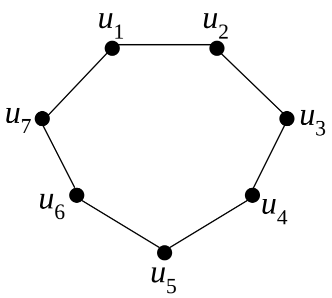


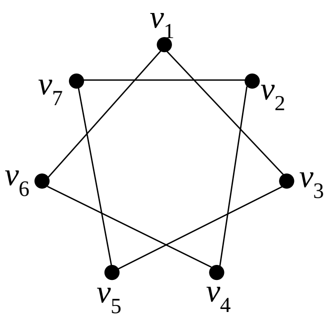
Remove vertex *d* from the graph.



Remove edge {*c*, *d*} from the graph.

3. (40 points) 下列兩個圖是否為同構 (isomorphic)? 如果它們是同構，寫出其相鄰矩陣 (adjacent matrices)；否則，解釋其非同構的理由。(Determine whether the given pair of graphs are isomorphic. If they are isomorphic, write their adjacent matrices; otherwise, explain the reason why.)





Both graphs have seven vertices and seven edges and each vertex is of degree two. We will try to find the isomorphism *f* of the two graphs. First, let us map vertex u_1 to v_1, $f(u_1)=v_1$. Then, we map their adjacent vertices, say, $f(u_2)=v_6$ and $f(u_7)=v_3$. Continually, we follow the adjacent vertices and obtain $f(u_3)=v_4$, $f(u_4)=v_2$, $f(u_5)=v_7$, $f(u_6)=v_5$. We already have $f(u_7)=v_3$. Hence, the two graphs *may be* isomorphic. To show they are indeed isomorphic, we will construct their corresponding adjacency matrices as the below:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and } \begin{array}{c} \\ v_1 \\ v_6 \\ v_4 \\ v_2 \\ v_7 \\ v_5 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_6 & v_4 & v_2 & v_7 & v_5 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Since the two adjacency matrices are identical, it implies the two graphs are isomorphic.