一〇五學年度 第一學期 期末考試
離散數學 資訊二甲、資訊二乙
學號：　　　　　　　　　　姓名：**Solutions**

1. (20 points) 使用數學歸納法證明 $1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$，$n$ 是一個正整數。 (Use mathematical induction to prove that $1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$, where n is a positive integer.)

Let $P(n)$ be the statement $1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$, where n is a positive integer.
Basis step: Since $1\times1!=1=(1+1)!-1$, $P(1)$ is true.
Inductive step:
Inductive hypothesis: Assume for all k with $k\geq1$, $P(k)$: $1\times1!+2\times2!+\ldots+k\times k!=(k+1)!-1$, is true.
We show the following steps:

$$\sum_{i=1}^{k+1} i\times i! = \left(\sum_{i=1}^{k} i\times i!\right) + (k+1)(k+1)!$$
$$= \big((k+1)!-1\big) + (k+1)(k+1)! = (k+1)!\,\big(1+(k+1)\big)-1$$
$$= (k+1)!\,(k+2)-1 = (k+2)!-1 = \big((k+1)+1\big)!-1.$$

Hence $P(k+1)$ is also true.

By mathematical induction, $P(n)$, i.e., $1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$, is true for all positive integer n.

2. (20 points) (a) 定義「$f(x)$ 是 $\Theta(g(x))$」。(Define "$f(x)$ is $\Theta(g(x))$.")
(b) 若 $f_1(x)$ 與 $f_2(x)$ 為整數對應到實數的函數，假設 $f_1(x)$ 是 $\Theta(g_1(x))$ 以及 $f_2(x)$ 是 $\Theta(g_2(x))$。證明 $(f_1 f_2)(x)$ is $\Theta(g_1g_2(x))$。 (Let $f_1(x)$ and $f_2(x)$ be functions from the set of positive integers to the set of real numbers. Suppose $f_1(x)$ is $\Theta(g_1(x))$ and $f_2(x)$ is $\Theta(g_2(x))$. Prove $(f_1 f_2)(x)$ is $\Theta(g_1g_2(x))$.)

(a) Function $f(x)$ is $\Theta(g(x))$, if there are real numbers C_1 and C_2, and a positive real number k, such that $C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$, whenever $x>k$.
(b) From the definition of big-Θ notation, there are constants C_{11}, C_{12}, C_{21}, C_{22}, k_1, and k_2 such that
$C_{11}|g_1(x)|\leq|f_1(x)|\leq C_{12}|g_1(x)|$ when $x>k_1$ and
$C_{21}|g_2(x)|\leq|f_2(x)|\leq C_{22}|g_2(x)|$ when $x>k_2$.
When x is greater than $\max(k_1, k_2)$, it follows that
$|(f_1f_2)(x)| = |f_1(x)|\,|f_2(x)| \geq C_{11}|g_1(x)|\,C_{21}|g_2(x)| \geq C_{11}C_{21}\,|g_1g_2(x)| = C_1\,|g_1g_2(x)|$,
$|(f_1f_2)(x)| = |f_1(x)|\,|f_2(x)| \leq C_{12}|g_1(x)|\,C_{22}|g_2(x)| \leq C_{12}C_{22}\,|g_1g_2(x)| = C_2\,|g_1g_2(x)|$,

where $C_1=C_{11}C_{21}$ and $C_2=C_{12}C_{22}$. From this inequality, it follows that $f_1(x)\,f_2(x)$ is $\Theta(g_1(x)\,g_2(x))$.

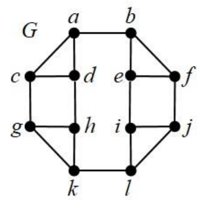
3. (20 points) 證明一個含有兩個頂點以上的簡單圖 (simple graph) 至少有兩個不同頂點 (vertex) 有相同 degree。(Prove that in a simple graph with at least two vertices there must be two distinct vertices that have the same degree.) (提示：使用鴿籠原理。Hint: Use the pigeonhole principle.)

Let $G=(V, E)$ be a simple graph and $|V|=n$. Since G is a simple graph, it has neither cycles nor multi-edges. Hence, the degree of a vertex v in G must be between 0 and $n-1$, i.e.,

$0 \le \deg(v) \le n-1$. However, if there exists a vertex with degree zero, all other vertices cannot have degree $n-1$. Namely, vertices of G cannot have degree zero and degree $n-1$ at the same time. Therefore, there are only $n-1$ choices of degrees for n vertices of G. By the pigeonhole principle, there must be at two vertices in V with the same degree.

4. (20 points) (a) 對於圖 G 定義 $\kappa(G)$ 和 $\lambda(G)$。(Define $\kappa(G)$ and $\lambda(G)$ for graph G.)

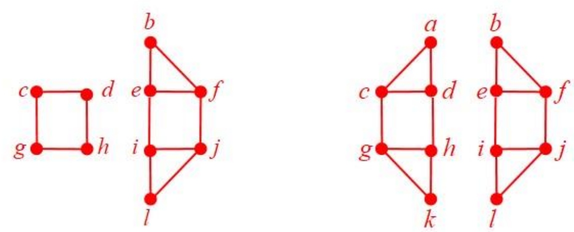
   (b) 考慮下圖 $G=(V, E)$, 寫出 G 的$\kappa(G)$, $\lambda(G)$ 和 $\min_{v\in V} \deg(v)$。(Consider graph $G=(V, E)$ as below. Write $\kappa(G)$, $\lambda(G)$, 和 $\min_{v\in V} \deg(v)$.)



   (c) 對任何的圖 $G=(V, E)$, $\kappa(G)$, $\lambda(G)$, 和 $\min_{v\in V} \deg(v)$ 之間有何關係? (For any graph $G=(V, E)$, what are the relations among $\kappa(G)$, $\lambda(G)$, and $\min_{v\in V} \deg(v)$?)

   (a) For graph G, $\kappa(G)$ is called vertex connectivity which is the minimum number of vertices being removed to disconnect G. For graph G, $\lambda(G)$ is called edge connectivity which is the minimum number of edges being removed to disconnect G.

   (b) Remove one vertex or one edge does not disconnect the graph. Removing vertices a and k, the graph becomes two disconnected components. Hence, $\kappa(G)=2$. Removing edges $\{a, b\}$ and $\{k, l\}$, the graph becomes two disconnected components. Hence, $\lambda(G)=2$. All vertices in G are of degree 3. Hence, $\min_{v\in V} \deg(v)=3$.

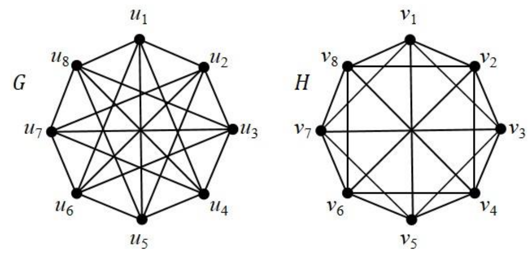


Remove vertices a and k from the graph.

Remove edges $\{a, b\}$ and $\{k, l\}$ from the graph.

   (c) For any graph G, $\kappa(G) \le \lambda(G) \le \min_{v\in V} \deg(v)$.

5. (20 points) 圖 $G = (V, E)$ 的補圖定義為 $\bar{G} = (V, V \times V - E)$。同時，有一個定理敘述：「兩個圖 G 和 H 是同構若且唯若它們的補圖 $\bar{G}$ 和 $\bar{H}$ 是同構。」判斷下列兩個圖是否為同構? 如果它們是同構，寫出其一對一對應函數及個別的相鄰矩陣；否則，寫出它們為何不同構的理由。(The complement of graph $G = (V, E)$ is defined as $\bar{G} = (V, V \times V - E)$. Also, we have a theorem that two graphs G and H are isomorphic if and only if their complements $\bar{G}$ and $\bar{H}$ are isomorphic. Determine whether the given pair of graphs are isomorphic. If they are isomorphic, show the one-to-one correspondence and their adjacency matrices; otherwise, give the reason(s) why they are not isomorphic.)



Consider the complements of G and H as shown below. Omitting cycles, since the complement graph $\bar{G}$ has two cycles of length 4 and the complement graph $\overline{H}$ has one cycle of length 8, they are not isomorphic. Hence, the graphs G and H given in the problem are not isomorphic.