一〇五學年度 第一學期 期中考試
離散數學 資訊二甲、資訊二乙
學號：　　　　　　　　　姓名：**Solutions**

1. (10 points) 寫出下列句子的***否定句***，並簡化所得結果的邏輯式至所有否定符號只出現在命題或述句符號之前。寫出簡化的步驟。(***Negate*** the following statement and ***simplify*** each resulting statement to place the negation operator appearing only before propositional or predicate symbols. Show simplification steps.)

   $\exists x\colon \forall y\colon ((m(x, y) \wedge \neg n(y, x)) \rightarrow \forall z\colon r(x, y, z))$

   $\neg(\exists x\colon \forall y\colon ((m(x, y) \wedge \neg n(y, x)) \rightarrow \forall z\colon r(x, y, z)))$

   $\equiv \neg(\exists x\colon \forall y\colon (\neg(m(x, y) \wedge \neg n(y, x)) \vee \forall z\colon r(x, y, z)))$

   $\equiv \forall x\colon \neg(\forall y\colon (\neg(m(x, y) \wedge \neg n(y, x)) \vee \forall z\colon r(x, y, z)))$

   $\equiv \forall x\colon \exists y\colon (\neg(\neg(m(x, y) \wedge \neg n(y, x)) \vee \forall z\colon r(x, y, z)))$

   $\equiv \forall x\colon \exists y\colon (\neg\neg(m(x, y) \wedge \neg n(y, x)) \wedge \neg\forall z\colon r(x, y, z))$

   $\equiv \forall x\colon \exists y\colon (m(x, y) \wedge \neg n(y, x) \wedge \neg\forall z\colon r(x, y, z))$

   $\equiv \forall x\colon \exists y\colon (m(x, y) \wedge \neg n(y, x) \wedge \exists z\colon \neg r(x, y, z))$

2. (20 points) A, B, 和 C 為三個集合，證明 $(A - B) \cup (A - C) = A - (B \cap C)$. 使用此指定之證明方法 $x \in (A - B) \cup (A - C) \Leftrightarrow x \in A - (B \cap C)$. (Let A, B, and C be sets. Show that $(A - B) \cup (A - C) = A - (B \cap C)$. Write proof steps and reasons. Use the designated proving method $x \in (A - B) \cup (A - C) \Leftrightarrow x \in A - (B \cap C)$.

$x \in (A - B) \cup (A - C)$	
$\Leftrightarrow x \in (A - B) \vee x \in (A - C)$	(Definition of union)
$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$	(Definition of difference)
$\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C)$	(Distributed law of $\wedge$ over $\vee$)
$\Leftrightarrow x \in A \wedge (\neg x \in B \vee \neg x \in C)$	(Definition of not belonging to)
$\Leftrightarrow x \in A \wedge \neg(x \in B \wedge x \in C)$	(De Morgan's Law)
$\Leftrightarrow x \in A \wedge \neg(x \in B \cap C)$	(Definition of intersection)
$\Leftrightarrow x \in A \wedge x \notin B \cap C$	(Definition of not belonging to)
$\Leftrightarrow x \in A - (B \cap C)$	(Definition of difference)
$\therefore (A - B) \cup (A - C) = A - (B \cap C)$.	

3. (20 points) f, g, 和 h 都是由 $\mathbf{Z}$ 對應到 $\mathbf{Z}$ 的函數，且其定義為：$f(x) = x-10$, $g(x) = x^2+2$, 和 $h(x) = x^3-1$，回答以下的問題。(Let f, g, and h be $\mathbf{Z}$ to $\mathbf{Z}$ functions defined as: $f(x) = x-10$, $g(x) = x^2+2$, and $h(x) = x^3-1$. Answer the following questions.)
   (a) 以上的函數何者為 ***injection***? 何者為 ***surjection***? 何者為 ***bijection***? (Which of the above functions is an ***injection***? Which one is a ***surjection***? Which one is a ***bijection***?
   (b) 寫出 $g \circ f$ 及 $h \circ g$. (Write $g \circ f$ and $h \circ g$.)
   (a) Functions f and h are injections (one-to-one functions). Functions f is a surjection (onto function). Function f is a bijection (one-to-one correspondence).
   (b) $g \circ f(x) = g(f(x)) = g(x-10) = (x-10)^2+2 = x^2-20x+100+2 = x^2-20x+102$.

   $h \circ g(x) = h(g(x)) = h(x^2+2) = (x^2+2)^3-1 = x^6+6x^4+12x^2+8-1 = x^6+6x^4+12x^2+7$.

4. (30 points) 假設集合 $A=\{1, 2, 3, 4, 5, 6, 7\}$ 和對等關係 $R=\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 6), (4, 3), (4, 4), (4, 6), (5, 5) (6, 3), (6, 4), (6, 6), (7, 7)\}$。回答下列問題：(Suppose

$A=\{1, 2, 3, 4, 5, 6, 7\}$ and equivalence relation $R=\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 6), (4, 3), (4, 4), (4, 6), (5, 5) (6, 3), (6, 4), (6, 6), (7, 7)\}$. Answer the following questions:)

(a) 反身律的條件是 $\forall x\in A$: $(x, x)\in R$；多少個 R 的元素對 (pair) 滿足此條件? 解釋你的答案。(The condition of reflexive law is $\forall x\in A$: $(x, x)\in R$. How many pairs in R satisfy this condition? Explain your answer.)

(b) 遞移律的條件是 $\forall x, y, z\in A$: $(x, y)\in R\wedge(y, z)\in R\Rightarrow(x, z)\in R$；多少組 R 的元素對 (pair) 滿足此條件? 解釋你的答案。(The condition of transitive law is $\forall x, y, z\in A$: $(x, y)\in R\wedge(y, z)\in R\Rightarrow(x, z)\in R$. How many groups of pairs in R satisfy this condition? Explain your answer.)

(c) A 有幾個對等類別 (equivalence class) $[x]_R$? 寫出所有 A 的對等類別。(How many equivalence classes $[x]_R$ of A? Write all equivalence classes of A.

(a) For every element $x\in A$, we must have $(x, x)\in R$ satisfying $(x, x)\in R$. Since because there are seven elements in A, there are seven pairs of (x, x) in R.

(b) (i) Consider $\{1, 2\}\subseteq A$. Since for each x, y, and z, we have two choices 1 and 2, there are $2^3=8$ groups of pairs satisfying $(x, y)\in R\wedge(y, z)\in R\Rightarrow(x, z)\in R$. (ii) Consider $\{3, 4, 6\}\subseteq A$. Since for each x, y, and z, we have three choices 3, 4, and 6, there are $3^3=27$ groups of pairs satisfying $(x, y)\in R\wedge(y, z)\in R\Rightarrow(x, z)\in R$. (ii) Consider $\{5\}\subseteq A$ and $\{7\}\subseteq A$. In each case, since for each x, y, and z, we have one choice 5 or 7, there is $1^3=1$ group of pair satisfying $(x, y)\in R\wedge(y, z)\in R\Rightarrow(x, z)\in R$. The total number of groups is 8+27+1+1=37.

(c) There are four equivalence classes of A. The equivalence classes are $[1]_R=[2]_R=\{1, 2\}$, $[3]_R=[4]_R=[6]_R=\{3, 4, 6\}$, $[5]_R=\{5\}$, and $[7]_R=\{7\}$.

5. (20 points) 使用數學歸納法證明對所有的自然數 $n\in\mathbf{N}$, $n\geq0$, $n^3+(n+1)^3+(n+2)^3$ 可被 9 整除。(Use mathematical induction to prove for all natural numbers $n\in\mathbf{N}$, $n\geq0$, $n^3+(n+1)^3+(n+2)^3$ is divisible by 9.

Basis step: When $n=0$, $0^3+(0+1)^3+(0+2)^3=1+8=9$ is divisible by 9.

Inductive step:

Inductive hypothesis, Assume $k^3+(k+1)^3+(k+2)^3$ is divisible by 9.

$(k+1)^3+((k+1)+1)^3+((k+1)+2)^3$

$= (k+1)^3+(k+2)^3+(k+3)^3$

$= (k+1)^3+(k+2)^3+k^3+9k^2+27k+27$

$= k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$

By inductive hypothesis, we have $k^3+(k+1)^3+(k+2)^3$ is divisible by 9, and $9(k^2+3k+3)$ is also divisible by 9. Hence, $(k+1)^3+((k+1)+1)^3+((k+1)+2)^3=k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$ is divisible by 9.

By mathematical induction, we have $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all $n\in\mathbf{N}$.