Homework Assignment #2
Solutions

1. Determine whether each of these statements is true or false::
   (a) $0\in\varnothing$ (*false*) (b) $\varnothing\in\{0\}$ (*false*) (c) $\{0\}\subset\varnothing$ (*false*)
   (d) $\varnothing\subset\{0\}$ (*true*) (e) $x\in\{x\}$ (*true*) (f) $\{x\}\subseteq\{x\}$ (*true*)
   (g) $\{x\}\in\{x\}$ (*false*) (h) $\{x\}\in\{\{x\}\}$ (*true*) (i) $\varnothing\subseteq\{x\}$ (*true*)

2. Determine whether each of these sets is the power set of a set, where *a* and *b* are distinct elements:
   (a) $\varnothing$ (b) $\{\varnothing, \{a\}\}$ (c) $\{\varnothing, \{a\}, \{\varnothing, a\}\}$
   (d) $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
   (a) Yes, $\varnothing$ is the power set of itself, i.e., $\varnothing$.
   (b) Yes, $\{\varnothing, \{a\}\}$ is the power set of $\{\{a\}\}$.
   (c) Not.$\{\varnothing, \{a\}, \{\varnothing, a\}\}$ is not a power set of a set. If it is a power set of some set, say A, and $\{\varnothing, a\}$ is an element of $P(A)$, A must have $\varnothing$ as its element that implies $\{\varnothing\}\in P(A)$. However, $\{\varnothing\}\notin\{\varnothing, \{a\}, \{\varnothing, a\}\}$.
   (d) Yes, $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ is the power set of $\{a, b\}$.

3. Let $A=\{a, b, c\}$, $B=\{x, y\}$, and $C=\{0, 1\}$. Find the following Cartesian products:
   (a) $A\times B\times C$ (b) $C\times B\times A$ (c) $B\times B\times B$
   (a) $A\times B\times C=\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$.
   (b) $C\times B\times A=\{(0, x, a)$ $(0, x, b), (0, x, c), (0, y, a)$ $(0, x, b), (0, y, c), (1, x, a)$ $(1, x, b), (1, x, c), (1, y, a)$ $(1, x, b), (01, y, c)\}$.
   (c) $B\times B\times B=\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$.

4. Let A and B be sets. Show that
   (a) $(A\cap B)\subseteq A$ (b) $A-B\subseteq A$ (c) $A\cup(B-A)=A\cup B$
   (a) If $x\in A\cap B$, then $x\in A$ and $x\in B$. Hence, we have the fact that if $x\in A\cap B$, then $x\in A$, i.e., $(A\cap B)\subseteq A$.
   (b) If $x\in A-B$, then $x\in A$ and $x\notin B$. Hence, we have the fact that if $x\in A-B$, then $x\in A$, i.e., $A-B\subseteq A$.
   (c) (i) If $x\in A\cup(B-A)$, then $x\in A$ or $x\in B-A$. ($x\in A$ or $x\in B-A$) implies ($x\in A$ or $x\in B$), i.e., $x\in B\cup A$. Hence, we have the fact that if $x\in A\cup(B-A)$, then $x\in A\cup B$, i.e., $A\cup(B-A)\subseteq A\cup B$.
   (ii) For any element x, either $x\in A$ or $x\notin A$. Since $x\in A\cup B$, $x\in A$ or $x\in B$, if $x\notin A$, it must be $x\in B$, i.e., $x\in B-A$. Therefore, $x\in A\cup(B-A)$. We have $A\cup B\subseteq A\cup(B-A)$.
   By (i) and (ii), we conclude $A\cup(B-A)=A\cup B$.

5. Let A, B, and C be sets. Show that
   (a) $(A\cap B\cap C)\subseteq(A\cap B)$ (b) $(A-C)\cap(C-B)=\varnothing$
   (a) $x\in A\cap B\cap C$ $\Rightarrow x\in(A\cap B)\cap C$ (associative law of intersection)
   $\Rightarrow x\in A\cap B$ (definition of intersection)
   $\therefore$ $(A\cap B\cap C)\subseteq(A\cap B)$
   (b) Assume there exist an element x such that $x\in(A-C)\cap(C-B)$.
   $x\in(A-C)\cap(C-B)$
   $\Rightarrow x\in A-C \wedge x\in C-B$ (definition of intersection)

$\Rightarrow (x\in A \wedge x\notin C) \wedge (x\in C \wedge x\notin B)$ (definition of difference)
$\Rightarrow x\in A \wedge (x\notin C \wedge x\in C) \wedge x\notin B$ (associative law of conjunction)
$\Rightarrow x\notin C \wedge x\in C$ (definition of conjunction)
$\Rightarrow$ *false* (negation law)

Hence, the assumption "there exist an element x such that $x\in(A-C)\cap(C-B)$" does not hold. $\therefore (A-C)\cap(C-B)=\varnothing$.

6. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
(a) The function that assigns to each nonnegative integer its last digit.
(b) The function that assigns to a bit string the number of one bits in the string.
(a) Domain: N, range: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
(b) Domain: N, range: N.

7. Determine each of these functions from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ is one-to-one.
(a) $f(n)=n-1$ (b) $f(n) = n^2+1$ (c) $f(n)=n^3$
(a) $f(n)=n-1$ is a one-to-one function.
(b) $f(n) = n^2+1$ is not a one-to-one function.
(c) $f(n)=n^3$ is a one-to-one function.

8. (Rosen, 7th ed., Section 2.3, Exercise 14) Give an example of a function from $\boldsymbol{N}$ to $\boldsymbol{N}$ that is
(a) One-to-one but not onto.
(b) Onto but not one-to-one.
(c) Both onto and one-to-one.
(d) Neither one-to-one nor onto.
(a) Function $f(n) = n^2$ is a one-to-one but not onto function from $\boldsymbol{N}$ to $\boldsymbol{N}$.
(b) Function $f(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n \neq 0 \end{cases}$ is an onto but not one-to-one function from $\boldsymbol{N}$ to $\boldsymbol{N}$.
(c) Function $f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$ is a one-to-one and onto function from $\boldsymbol{N}$ to $\boldsymbol{N}$.
(d) Function $f(n) = n^2 - 3n + 2$ is a neither one-to-one nor onto function from $\boldsymbol{N}$ to $\boldsymbol{N}$.

9. Find $f\circ g$ and $g\circ f$, where $f(x)=x^2+1$ and $g(x)=x+2$ are functions form $\boldsymbol{R}$ to $\boldsymbol{R}$.
$f\circ g(x) = f(g(x)) = f(x+2)+1 = (x+2)^2+1 = x^2+4x+5.$
$g\circ f(x) = g(f(x)) = g(x^2+1) = (x^2+1)+2 = x^2+3.$

10. Let $f(x)=ax+b$ and $g(x)=cx+d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f\circ g=g\circ f$.
$f\circ g(x) = f(cx+d) = a(cx+d)+b = acx+ad+b,$
$g\circ f(x) = g(ax+b) = c(ax+b)+d = acx+bc+d.$
Hence, the necessary and sufficient condition of $f\circ g = g\circ f$ is $ad + b = bc + d$.

11. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals:
(a) $(n+1)^{n+1}$? (b) $\lfloor n/2\rfloor+\lceil n/2\rceil$?
(a) $a_0=(0+1)^{0+1}=1^1=1,$
$a_1=(1+1)^{1+1}=2^2=4,$
$a_2=(2+1)^{2+1}=3^3=27,$

$a_3=(3+1)^{3+1}=4^4=256.$

(b) $a_0=\lfloor 0/2\rfloor+\lceil 0/2\rceil=0+0=0,$

$a_1=\lfloor 1/2\rfloor+\lceil 1/2\rceil=0+1=1,$

$a_2=\lfloor 2/2\rfloor+\lceil 2/2\rceil=1+1=2,$

$a_3=\lfloor 3/2\rfloor+\lceil 3/2\rceil=1+2=3.$

12. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions:

(a) $a_n = a_{n-1}+3\,a_{n-2}$, $a_0=1$, $a_1=2$ (b) $a_n = n\,a_{n-1}+ n^2\,a_{n-2}$, $a_0=1$, $a_1=1$

(a) $a_0=1$, $a_1=2$,

$a_2=a_1+3a_0=2+3\times1=5,$

$a_3=a_2+3a_1=5+3\times2=11,$

$a_4=a_3+3a_2=11+3\times5=26.$

(b) $a_0=1$, $a_1=1$,

$a_2=2a_1+2^2a_0=2\times1+2^2\times1=6,$

$a_3=3a_2+3^2a_1=3\times6+3^2\times1=27,$

$a_4=4a_3+4^2a_2=4\times27+4^2\times6=204.$

13. An employee joined a company in 2009 with a starting salary of \$50,000. Every year this employee receives a raise of \$1,000 plus 5% of the salary of the previous year.

(a) Set up a recurrence relation for the salary of this employee n years after 2009.

(b) What will the salary of this employee be in 2017?

(c) Find an explicit formula for the salary of this employee n years after 2009.

(a) Recurrence relation: $a_0=50000$, $a_n=1.05\times a_{n-1}+1000$, for $n>0$.

(b) The year of 2017 is eight years after 2009. Hence, the salary in 2017 of this employee is a_8.

$a_0=50000,$

$a_1=1.05\times a_0+1000=1.05\times50000+1000=53500,$

$a_2=1.05\times a_1+1000=1.05\times53500+1000=57175,$

$a_3=1.05\times a_2+1000=1.05\times57175+1000=61033.75\approx61034,$

$a_4=1.05\times a_3+1000=1.05\times61034+1000=65085.70\approx65086,$

$a_5=1.05\times a_4+1000=1.05\times65086+1000=69340.30\approx69340,$

$a_6=1.05\times a_5+1000=1.05\times69340+1000=73807,$

$a_7=1.05\times a_6+1000=1.05\times73806+1000=78497.35\approx78497,$

$a_8=1.05\times a_7+1000=1.05\times78497+1000=83421.85\approx83422.$

(c) $a_0 = 50000$

$a_1 = 1.05\times50000 + 1000$

$a_2 = 1.05\times a_1 + 1000 = 1.05\times(1.05\times50000 + 1000) + 1000$
$= 1.05^2\times50000 + 1.05\times1000 + 1000$

$a_3 = 1.05\times a_2 + 1000$
$= 1.05\times(1.05^2\times50000 + 1.05\times1000 + 1000) + 1000$
$= 1.05^3\times50000 + 1.05^2\times1000 + 1.05\times1000 + 1000$

…

$$a_n = 1.05^n\times50000 + 1000 \times \sum_{k=0}^{n-1}(1.05^k)$$

$$\therefore a_8 = 1.05^8 \times 50000 + \sum_{k=0}^{7} (1.05^k) = 83421.88 \approx 83422.$$

14. What are values of these sums, where $S=\{1, 3, 5, 7\}$?

(a) $\sum_{j \in S} j^2$ (b) $\sum_{j \in S} (1/j)$

(a) $\sum_{j \in S} j^2 = \sum_{j \in \{1,3,5,7\}} j^2 = 1^2 + 3^2 + 5^2 + 7^2 = 1 + 9 + 25 + 49 = 84.$

(b) $\sum_{j \in S} {}^1\!/_j = \sum_{j \in \{1,3,5,7\}} \left({}^1\!/_j\right) = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{105+35+21+15}{105} = \frac{176}{105}.$

15. Compute each of these double sums.

(a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j)$ (b) $\sum_{i=0}^{2} \sum_{j=1}^{3} i\,j$

(a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j) = \sum_{i=1}^{2} \big((i + 1) + (i + 2) + (i + 3)\big) = \sum_{i=1}^{2} (3i + 6)$
$= (3 \times 1 + 6) + (3 \times 2 + 6) = 9 + 12 = 21.$

(b) $\sum_{i=1}^{2} \sum_{j=1}^{3} ij = \sum_{i=1}^{2} (i + 2i + 3i) = \sum_{i=1}^{2} 6i = 6 \times 1 + 6 \times 2 = 18.$

16. Find $\sum_{k=99}^{200} k^3$.

Recall that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

$$\sum_{k=99}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3$$

$$= \frac{200^2 \times (200 + 1)^2}{4} - \frac{98^2 \times (98 + 1)^2}{4}$$

$$= \frac{200^2 \times 201^2}{4} - \frac{98^2 \times 99^2}{4}$$

$$= \frac{1616040000 - 94128804}{4}$$

$$= \frac{1521911196}{4}$$

$$= 380477799.$$