Homework Assignment #3
Solutions

1. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
   (a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
   (b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
   (c) {(2, 4), (4, 2)}
   (d) {(1, 2), (2, 3), (3, 4) }
   (e) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
   (a) Transitive
   (b) Reflexive, symmetric, and transitive
   (c) Symmetric
   (d) Antisymmetric
   (e) None of these properties

2. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y)\in R$, if and only if
   (a) $x \neq y$ (b) $x = y + 1$ or $x = y - 1$ (c) x is a multiple of y
   (d) x and y are both negative or both nonnegative (e) $x \geq y^2$
   (a) Symmetric (b) Symmetric (c) Reflexive, antisymmetric, transitive
   (d) Reflexive, symmetric, transitive (e) Antisymmetric, transitive

3. How many nonzero entries does the matrix representing the relation R on A={1, 2, 3, …, 1000} consisting of the first 1000 positive integers have if R is
   (a) $\{(a, b) \mid a \leq b\}$? (b) $\{(a, b) \mid a = b \pm 1\}$? (c) $\{(a, b) \mid a + b = 1001\}$?
   Note that The total number of entries in the matrix is $1{,}000^2=1{,}000{,}000$.
   (a) When a = 1, row 1 of the matrix has 1,000 entries of 1, when hen a = 2, row 2 of the matrix has 9 entries of 1. When $a = i$, row i of the matrix has $1{,}000-i+1$ entries of 1. That is, all matrix entries on the diagonal and the upper triangular are 1. The total number of these entries is $1 + 2 + \ldots + 1000 = \sum_{i=1}^{1000} i = \frac{(1000+1)1000}{2} = 500{,}500$.
   (b) There are two entries of 1's in each row except the first and the last row. Each of the first row and the last row has one entry of 1. Hence, the total number of 1 entries is 998×2+2=1998.
   (c) The entries one the left of the anti-diagonal elements are 1, i.e., (1, 1000), (2, 999), …, (1000, 1) have entries 1. There are 1000 entries of 1.

4. Answer the following two questions:
   (a) How can the matrix for $\bar{R}$, the complement of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?
   (b) How can the matrix for R^{-1}, the inverse of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?
   (a) If the matrix representing relation R is M, the matrix representing $\bar{R}$, the complement of the relation R, is matrix $\bar{M}$ that change the entry of 1 in M to 0, and the entry of 0 in M to 1.
   (b) If the matrix representing relation R is M, the matrix representing R^{-1}, the inverse of the relation R, is the transpose of M^T, the transpose matrix of M.

5. If relation *R* is represented as a directed graph, explain how to determine whether *R* is (a) reflexive, (b) symmetric, (c) antisymmetric, and (d) transitive using the directed graph?

(a) *R* is reflexive, if every vertex of the directed graph has a loop.
(b) *R* is symmetric, if any two connected vertices have a cycle of length two.
(c) *R* is antisymmetric, if any two vertices have no cycle of length 2.
(d) *R* is transitive, if any two vertices with a path of length 2 are connected in the same direction.