Homework Assignment #4
Solutions

1. Let $P(n)$ be the statement $1^3+2^3+\ldots+n^3=(n(n+1)/2)^2$ for the positive integer n. Prove $P(n)$ is true.

   Basis step: We show that $\sum_{i=1}^{1} i^3 = 1^3 = 1 = (1(1+1)/2)^2$. Hence, $P(1)$ is true.

   Inductive step:

   Inductive hypothesis: Assume for all k with $k\geq 1$, $P(k)$: $\sum_{i=1}^{k} i^3 = (k(k+1)/2)^2$ is true.

   We show the following steps:

   $$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 = (k(k+1)/2)^2 + (k+1)^3 = \frac{k^2(k+1)^2+4(k+1)^3}{4} = \frac{(k+1)^2\left(k^2+4(k+1)\right)}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

   Hence, $P(k+1)$ is true.

   By mathematical induction, $P(n)$, i.e., $1^3+2^3+\ldots+n^3=(n(n+1)/2)^2$, is true for all positive integer n.

2. Prove that $1\bullet 1!+2\bullet 2!+\ldots+n\bullet n!=(n+1)!-1$ whenever n is a positive integer.

   Let $P(n)$ be the statement $1\bullet 1!+2\bullet 2!+\ldots+n\bullet n!=(n+1)!-1$, where n is a positive integer.

   Basis step: Since $1\bullet 1!=1=(1+1)!-1$, $P(1)$ is true.

   Inductive step:

   Inductive hypothesis: Assume for all k with $k\geq 1$, $P(k)$: $1\bullet 1!+2\bullet 2!+\ldots+k\bullet k! = (k+1)!-1$ is true.

   We show the following steps:

   $$\sum_{i=1}^{k+1} i \times i! = \left(\sum_{i=1}^{k} i \times i!\right) + (k+1)(k+1)!$$
   $$= \left((k+1)!-1\right) + (k+1)(k+1)! = (k+1)!\left(1+(k+1)\right) - 1$$
   $$= (k+1)!\,(k+2) - 1 = (k+2)! - 1 = \left((k+1)+1\right)! - 1.$$

   Hence $P(k+1)$ is also true.

   By mathematical induction, $P(n)$, i.e., $1\bullet 1!+2\bullet 2!+\ldots+n\bullet n!=(n+1)!-1$, is true for all positive integer n.

3. Prove that $2-2\bullet 7+2\bullet 7^2-\ldots+2(-7)^n = (1-(-7)^{n+1})/4$ whenever n is a nonnegative integer.

   Let $P(n)$ be the statement $2-2\bullet 7+2\bullet 7^2-\ldots+2(-7)^n = (1-(-7)^{n+1})/4$, where n is a nonnegative integer.

   Basis step: Since $2(-7)^0=2=8/4=(1-(-7)^{0+1})/4$, $P(0)$ is true.

   Inductive step:

   Inductive hypothesis: Assume for all k with $k\geq 0$, $P(k)$: $2-2\bullet 7+2\bullet 7^2-\ldots+2(-7)^k = (1-(-7)^{k+1})/4$ is true.

   We show the following steps:

   $$\sum_{i=0}^{k+1} 2\times(-7)^i = \left(\sum_{i=0}^{k} 2\times(-7)^i\right) + 2\times(-7)^{k+1}$$
   $$= \frac{1-(-7)^{k+1}}{4} + 2\times(-7)^{k+1} = \frac{1-(-7)^{k+1} + 8\times(-7)^{k+1}}{4}$$

$= \frac{1+7\times(-7)^{k+1}}{4} = \frac{1-(-7)\times(-7)^{k+1}}{4} = \frac{1-(-7)^{(k+1)+1}}{4}.$

Hence $P(k+1)$ is also true.

By mathematical induction, $P(n)$, i.e., $2-2\cdot 7+2\cdot 7^2-\ldots+2(-7)^n = (1-(-7)^{n+1})/4$ whenever n is a nonnegative integer.

4. Prove that $\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$ whenever n is a nonnegative integer.

Let $P(n)$ be the statement $\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$, where n is a nonnegative integer.

Basis *step*: Since $\sum_{j=0}^{0}\left(-\frac{1}{2}\right)^j = \left(-\frac{1}{2}\right)^0 = 1 = \frac{2^{0+1}+(-1)^0}{3\cdot 2^0}$, $P(0)$ is true.

Inductive step:

Inductive hypothesis: Assume $\sum_{j=0}^{k}\left(-\frac{1}{2}\right)^j = \frac{2^{k+1}+(-1)^k}{3\cdot 2^k}$ for all k with $k\geq 0$, is true.

We show the following steps:

$$\sum_{j=0}^{k+1}\left(-\frac{1}{2}\right)^j = \left(\sum_{j=0}^{k}\left(-\frac{1}{2}\right)^j\right) + \left(-\frac{1}{2}\right)^{k+1}$$
$$= \frac{2^{k+1}+(-1)^k}{3\cdot 2^k} + \left(-\frac{1}{2}\right)^{k+1}$$
$$= \frac{2^{k+1}+(-1)^k}{3\cdot 2^k} + \frac{(-1)^{k+1}}{2\cdot 2^k}$$
$$= \frac{2\cdot 2^{k+1}+2\cdot(-1)^k+3\cdot(-1)^{k+1}}{6\cdot 2^k}$$
$$= \frac{2^{k+2}-2\cdot(-1)\cdot(-1)^k+3\cdot(-1)^{k+1}}{3\cdot 2\cdot 2^k}$$
$$= \frac{2^{(k+1)+1}-2\cdot(-1)^{k+1}+3\cdot(-1)^{k+1}}{3\cdot 2^{k+1}}$$
$$= \frac{2^{(k+1)+1}+(-1)^{k+1}}{3\cdot 2^{k+1}}.$$

Hence $P(k+1)$ is also true.

By mathematical induction, $P(n)$, i.e., $\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$ whenever n is a nonnegative integer

5. Prove that for every positive integer n, $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)= n(n+1)(n+2)(n+3)/4$.

Let $P(n)$ be the statement $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)= n(n+1)(n+2)(n+3)/4$, where n is a positive integer.

Basis step: We show that $1\times(1+1)\times(1+2)=6=1\times(1+1)\times(1+2)\times(1+3)/4$. Hence, $P(1)$ is true.

Inductive step:

Inductive hypothesis: Assume for all k with $k\geq 1$, $P(k)$: $\sum_{i=1}^{k} i(i+1)(i+2) = k(k+1)(k+2)(k+3)/4$ is true.

We show the following steps:

$$\sum_{i=1}^{k+1} i(i+1)(i+2) = \left(\sum_{i=1}^{k} i(i+1)(i+2)\right) + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$
$$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$
$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$
$$= \frac{(k+1)\big((k+1)+1\big)\big((k+1)+2\big)\big((k+1)+3\big)}{4}.$$

Hence, $P(k+1)$ is true.

By mathematical induction, $P(n)$, i.e., $1\times2\times3+2\times3\times4+\ldots+n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$, is true for all positive integer n.

6. Let $P(n)$ be the statement that $n!<n^n$, where n is an integer greater than 1.
   a) What is the statement $P(2)$?
   b) Show that $P(2)$ is true, completing the basis step of the proof.
   c) What is the inductive hypothesis?
   d) What do you need to prove in the inductive step?
   e) Complete the inductive step.
   f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

   a) The statement $P(2)$ is $2!<2^2$.
   b) $P(2)$: $2!=1\times2=2<4=2^2$. Hence the basis step $P(2)$ is true.
   c) The inductive hypothesis is $P(k)$, $k!<k^k$, is true, for all k with $k\geq2$.
   d) We need to prove $(k+1)!<(k+1)^{k+1}$.
   e) $(k+1)! = k!(k+1) < k^k(k+1) < (k+1)^k(k+1) = (k+1)^{k+1}$. Hence, $P(k+1)$ is true.
   f) Steps a) to e) are the proof steps of mathematical induction with basis step starting from $n=2$ which asserts that this inequality, $P(n)$: $n!<n^n$, is true whenever n is an integer greater than 1.

7. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$, for $j=1, 2, \ldots, n$. Then
$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be sets such that $A_j \subseteq B_j$, for $j=1, 2, \ldots, n$, and $P(n)$ be the statement:
$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

Basis step: We show that $\bigcap_{j=1}^{1} A_j = A_1 \subseteq B_1 = \bigcap_{j=1}^{1} B_j$. Hence, $P(1)$ is true.

Inductive step:

Inductive hypothesis: Assume for all k with $k\geq1$, $P(k)$: $\bigcap_{j=1}^{k} A_j \subseteq \bigcap_{j=1}^{k} B_j$ is true.

We show the following steps:
$$\bigcap_{j=1}^{k+1} A_j = \left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1} \subseteq \left(\bigcap_{j=1}^{k} B_j\right) \cap A_{k+1} \subseteq \left(\bigcap_{j=1}^{k} B_j\right) \cap B_{k+1} = \bigcap_{j=1}^{k+1} B_j.$$

Hence, $P(k+1)$ is true.

By mathematical induction, $P(n)$: $\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j$ is true.

8. Prove that if A_1, A_2, …, A_n and B are sets then $(A_1 \cup A_2 \cup \ldots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$.

Let A_1, A_2, …, A_n and B_1, B_2, …, B_n be sets such that $A_j \subseteq B_j$, for $j=1, 2, \ldots, n$, and $P(n)$ be the statement $(A_1 \cup A_2 \cup \ldots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$.

Basis step: We know that $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$, by distributive law. Hence, $P(2)$ is true.

Inductive step:

Inductive hypothesis: Assume for all k with $k \geq 1$, $P(k)$: $(A_1 \cup A_2 \cup \ldots \cup A_k) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B)$ is true.

We show the following steps:

$(A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}) \cap B$

$= ((A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}) \cap B$

$= ((A_1 \cup A_2 \cup \ldots \cup A_k) \cap B) \cup (A_{k+1} \cap B)$

$= ((A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B)) \cup (A_{k+1} \cap B)$

$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B) \cup (A_{k+1} \cap B)$

By mathematical induction, $P(n)$: $(A_1 \cup A_2 \cup \ldots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$ is true.

9. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. Prove $P(n)$ is true for $n \geq 18$.

Let $P(n)$ be the statement: if n is a positive integer with $n \geq 18$, there exist nonnegative integers s and t such that $n=4s+7t$.

Basis step: We show the following basis cases:

$P(18)$: $18=1\times4+2\times7$,

$P(19)$: $19=3\times4+1\times7$,

$P(20)$: $20=5\times4+0\times7$,

$P(21)$: $21=0\times4+3\times7$.

Inductive step:

Inductive hypothesis: Assume that when $k \geq 21$, $P(j)$ is true, i.e., $j=4s+7t$, for some s and t, for $18 \leq j \leq k$.

We show $k+1=(k-3)+4=4s+7t+4=4(s+1)+7t$, for some s and t because $k \geq 21$ implies $18 \leq k-3 < k$.

By strong mathematical induction, $P(n)$ is true, i.e, if n is a positive integer with $n \geq 18$, there exist nonnegative integers s and t such that $n=4s+7t$

10. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, a sum of a subset of the integers $2^0=1$, $2^1=2$, $2^2=4$, and so on. (Hint: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. When it is even, note that $(k+1)/2$ is an integer.)

Let $P(n)$ be the statement: if n is a positive integer, n is written as a sum of $2^0=1$, $+2^1=2$, $2^2=4$, and so on.

Basis step: We have $1=2^0$. Hence $P(0)$ is true.

Inductive step:

Inductive hypothesis: *Assume* that when $k \geq 1$, $P(j)$ is true, i.e., j is written as the sum of $2^0=1$, $2^1=2$, $2^2=4$, and so on, for all j, where $1 \leq j \leq k$,.

We know that $k+1$ is either an even number of an odd number.

a) If $k+1$ is even, $k = 2 \times k/2$. By induction hypothesis, since $1 \leq k/2 \leq k$, we have $\frac{k}{2} = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}$ for some i_1, i_2, … i_r that $i_1 < i_2 < \ldots < i_r$. We obtain that $k+1 = 2\times\frac{k}{2} = 2\times(2^{i_1}+2^{i_2}+\cdots+2^{i_r}) = 2^{i_1+1}+2^{i_2+1}+\cdots+2^{i_r+1}$ and $i_1+1<i_2+1<\ldots<i_r+1$.

b) If $k+1$ is odd, $k = 1+2\times k/2$. By induction hypothesis, since $1\le k/2\le k$, we have $\frac{k}{2} = 2^{i_1}+2^{i_2}+\cdots+2^{i_r}$ for some $i_1, i_2, \ldots i_r$ that $i_1<i_2<\ldots<i_r$. We obtain that $k+1 = 1+2\times\frac{k}{2} = 2^0+2\times(2^{i_1}+2^{i_2}+\cdots+2^{i_r}) = 2^0+2^{i_1+1}+2^{i_2+1}+\cdots+2^{i_r+1}$ and $0<i_1+1<i_2+1<\ldots<i_r+1$.

We have shown that no matter whether $k+1$ is even or odd, it can be written as the sum of $2^0=1$, $2^1=2$, $2^2=4$, and so on.

By strong mathematical induction, we conclude that $P(n)$ is true for every positive integer n.

11. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0)=-1$, $f(1)=2$, and for $n=1, 2, \ldots$

a) $f(n+1) = f(n) + 3f(n-1)$.

b) $f(n+1) = f(n)^2 f(n-1)$.

a) $f(2) = f(1) + 3\,f(0) = 2+3\times(-1) = -1$,
$f(3) = f(2) + 3\,f(1) = -1+3\times2 = 5$,
$f(4) = f(3) + 3\,f(2) = 5+3\times(-1) = 2$,
$f(5) = f(4) + 3\,f(3) = 2+3\times5 = 17$.

b) $f(2) = f(1)^2\,f(0) = 2^2\times(-1) = -4$ (or -2^2),
$f(3) = f(2)^2\,f(1) = (-4)^2\times2 = 32$ (or 2^5),
$f(4) = f(3)^2\,f(2) = 32^2\times(-4) = -4096$ (or -2^{12}),
$f(5) = f(4)^2\,f(3) = (-4096)^2\times32 = 536870912$ (or 2^{29}).

12. Let f_n be the nth Fibonacci number. Prove that $f_1^2+f_2^2+\cdots+f_n^2 = f_nf_{n+1}$ where n is a positive integer.

Let $P(n)$ be the statement: $f_1^2+f_2^2+\cdots+f_n^2 = f_nf_{n+1}$ where n is a positive integer and f_i is the ith Fibonacci number, for $1\le i\le n+1$.

Recall the definition of Fibonacci number is:

$$f(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ f(n-1)+f(n-2) & \text{if } n>1 \end{cases}$$

Basis step: We have $f_1^2 = 1^2 = 1 = 1\times1 = f_1f_2 = f_1f_{1+1}$. Hence $P(1)$ is true.

Inductive step:

Inductive hypothesis: *Assume* that when $k\ge1$, $P(k)$ is true, i.e., $f_1^2+f_2^2+\cdots+f_k^2 = f_kf_{k+1}$.

We show that $f_1^2+f_2^2+\cdots+f_{k+1}^2 = f_1^2+f_2^2+\cdots+f_k^2+f_{k+1}^2 = f_kf_{k+1}+ +f_{k+1}^2 = f_{k+1}(f_k+f_{k+1}) = f_{k+1}f_{k+2}$, i.e., $P(k+1)$ is true

By mathematical induction, we conclude that $P(n)$, i.e., $f_1^2+f_2^2+\cdots+f_n^2 = f_nf_{n+1}$, is true for every positive integer n.