**Homework Assignment #6**
**Solutions**

1. How many positive integers between 100 and 999 inclusive
   a) have the same three decimal digits?
   b) are not divisible by either 3 or 4?
   c) are divisible by 3 but not by 4?
   d) are divisible by 3 and 4?
   a) The answer is 9 because the positive integers between 100 and 999 having the same three decimal digits are 111, 222, …, 999.

   There are 300 positive integers between 100 and 999 divisible by 3, 225 positive integers divisible by 4, and 75 positive integers divisible by both 3 and 4. From these results, we obtain the following answers:
   b) The answer is 999–99–(300+225–75)=450 positive integers between 100 and 999 not divisible by either 3 or 4.
   c) The answer is 300–75=225 positive integers between 100 and 999 divisible by 3 but not by 4.
   d) The answer is 75 positive integers between 100 and 999 divisible by 3 and 4.

2. How many bit strings of length seven either begin with two 0's or end with three 1's?
   For bit strings of length seven, there are $2^5$=32 bit strings beginning with two 0's, $2^4$=16 bit strings ending with three 1's, and $2^2$=4 bit strings both beginning with two 0' and ending with three 1's. Hence, the answer is 32+16–4=44.

3. How many bit strings of length 10 contain either five consecutive 0's ore five consecutive 1's?
   For bit strings of length 10, there are $2^5$=32 bit strings beginning with five consecutive 0's. If the first bit of the five consecutive 0's starts from the second to the sixth bit counting from the most significant bit, their preceding bit must be 1 and there are other bits can be either 0 or 1. Hence, there are $2^4\times5$=80 bit strings with five consecutive 0's which does not start from the most significant bit. Totally, the number of bit strings with five consecutive 0's is 32+80=112. Similarly, the he number of bit strings with five consecutive 1's is also 32+80=112. Furthermore, the two bit strings 0000011111 and 1111100000 are counted twice. Hence, the total number bit strings of length 10 contain either five consecutive 0's ore five consecutive 1's is 112+112–2=222.

4. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters. Also, there 32 reserved words in C.)
   There are 53 possible cases, uppercase letters, lowercase letters, and underscore for the first character. In this question, a variable of C programming language can be one to eight characters such that there can be zero to seven characters after the first one. Each of these characters has 63 possible cases, uppercase letters, lowercase letters, digits, or underscores. Hence, excluding the 32 reserved words, the number of all possible variables is

$$\left(53 \times \sum_{i=0}^{7} 63^i\right) - 32 = 53 \times \frac{63^8 - 1}{63 - 1} - 32$$
$$= 212{,}133{,}167{,}002{,}848 \approx 2.1 \times 10^{14}.$$

5. Suppose that $p$ and $q$ are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding $n$ that are relatively prime to $n$. (Hint: The principle of inclusion-exclusion is explained in Rosen, 7th ed., Section 6.1, pp. 382—384.)

   Let $P$ and $Q$ be the sets of integers in $\{1, 2, \ldots, n\}$ that are divisible by $p$ and $q$, respectively. The positive integers not exceeding $n$ that are relatively prime to $n$ are those integers not exceeding $n$ and not divisible by $p$ or $q$. Hence the number of positive integers not exceeding $n$ that are relatively prime to $n$ is $n-|P\cup Q|$. By the principle of inclusion-exclusion, $|P\cup Q|=|P|+|Q|-|P\cap Q|$. The number divisible by $p$ and $q$ in $\{1, 2, \ldots, n\}$ are $\lfloor n/p \rfloor$ and $\lfloor n/q \rfloor$, respectively. Clearly, $n$ is the only integer in $\{1, 2, \ldots, n\}$ divisible by $pq$, i.e., $|P\cap Q|=1$. Therefore, the number of positive integers not exceeding $n$ that are relatively prime to $n$ is

   $$n - (|P| + |Q| - |P\cap Q|) = n - (\lfloor n/p \rfloor + \lfloor n/q \rfloor - 1) = n - \lfloor n/p \rfloor - \lfloor n/q \rfloor + 1.$$

6. Let $(x_i, y_i)$, $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the $xy$ plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

   For a point with integer coordinates in $xy$ plane and $(x_i, y_i)$, $i = 1, 2, 3, 4, 5$, each of the five points is one of the four parity cases, (odd, odd), (odd, even), (even, odd), or (even, even). Since there are five points, $(x_i, y_i)$, $i = 1, 2, 3, 4, 5$, by the pigeonhole principle, there must be two points have the same parity. Let the two points with the same parity be $(x_i, y_i)$ and $(x_j, y_j)$. Then, we have the parity of $(x_i+x_j, x_i+x_j)$ is (even, even). Hence, the midpoint of these two points is $((x_i+x_j)/2, (x_i+x_j)/2)$ which has integer coordinates.

7. How many ordered pairs of integers $(a, b)$ are needed to guarantee that there are two ordered pairs $(a_1, b_1)$ and $(a_2, b_2)$ such that $a_1 \textbf{ mod } 5 = a_2 \textbf{ mod } 5$ and $b_1 \textbf{ mod } 5 = b_2 \textbf{ mod } 5$.

   Given a point $(a, b)$, $(a \textbf{ mod } 5, b \textbf{ mod } 5)$ is $(i, j)$, where $0\le i, j\le 4$. Totally, 25 possibly cases. Hence, by the pigeonhole principle, at least 26 ordered pairs of integer are needed to guarantee that there are two ordered pairs $(a_1, b_1)$ and $(a_2, b_2)$ such that $a_1 \textbf{ mod } 5 = a_2 \textbf{ mod } 5$ and $b_1 \textbf{ mod } 5 = b_2 \textbf{ mod } 5$.

8. A computer network consists of six computers. Each computer is directly connected at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

   Let $C(n)$ be the number of other computers connected by computer $n$, for $1\le n\le 6$. Since there are six computers, the values of $C(n)$, for $1\le n\le 6$, are 1, 2, 3, 4, or 5. By the pigeonhole principle, there must exist $i$ and $j$, $1\le i, j\le 6$ and $i\ne j$, $C(i)=C(j)$. That is, there are at least two computers in the network that are directly connected to the same number of other computers.

9. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four computers of the eight

computers, these four computers can directly access four different printers. Justify you answer.

Let the computers be $C_1$ to $C_8$ and the printers be $P_1$ to $P_4$. If we connect $C_i$ to $P_i$, for $1 \leq i \leq 4$, and connect $C_j$ to all printers, for $5 \leq j \leq 8$, then we need $4+4\times4=20$ cables. This connection satisfies the condition that any four computers can directly access four different printers. We will show it is impossible to use less than 20 cables. Assume 19 cables are used to connect eight computers to four printers. The average number of computers connected to a printer is 19/4 which is less than 5. Therefore, some printer must be connected to less than five computers, i.e., this printer is connected to four or less than 4 computers. Then, there are at least four computers not connected to it. Hence, it is not possible for these four computers to access four printers because they are directly connected to at most three printers. The least number of cables required is 20.

10. Prove that at a party where there are at least two people, there are two people who knows the same number of other people there. (Hint: The "knowing" relation is symmetric, i.e., if *a* knows *b*, then *b* also knows *a*.)

Suppose there are $n$ people in the party. Let $K(a)$ be the number of people that $a$ knows in the party. Then, the value of $K(a)$ must be one of the numbers 0, 1, 2, …, $n-1$. However, since the "knowing" relation is symmetric, if one person in the party knows no other people in the party, then it is impossible someone in the party knows all other $n-1$ people. Therefore, 0 and $n-1$ cannot be the value of $K(a)$ for all people $a$ in the party. Since there are only $n-1$ choices for $n$ people, there must be at least two people who knows the same number of other people in the party.