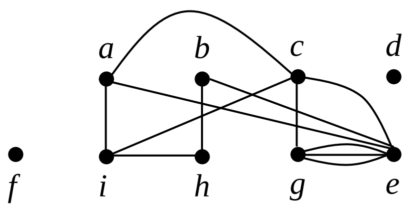
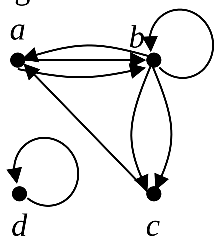
Homework Assignment #7
Solutions

1. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices



The number of vertices, $|V|$, is 9.
The number of edges, $|E|$, is 12.
The degrees are $\deg(a)=3$, $\deg(b)=2$, $\deg(c)=4$, $\deg(d)=0$, $\deg(e)=6$, $\deg(f)=0$, $\deg(g)=4$, $\deg(h)=2$, and $\deg(i)=3$.
The isolated vertices are *d* and *f*. The graph has no pendant vertex.

2. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

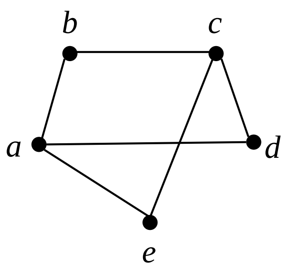


The number of vertices, $|V|$, is 4.
The number of edges, $|E|$, is 8.
The in-degrees and out-degrees are $\deg^-(a)=2$, $\deg^+(a)=2$, $\deg^-(b)=3$, $\deg^+(b)=4$, $\deg^-(c)=2$, $\deg^+(c)=1$, $\deg^-(d)=1$, $\deg^+(d)=1$.

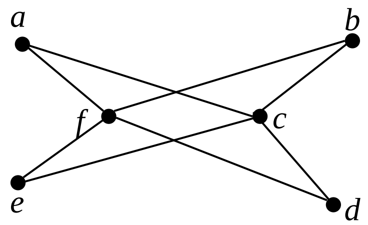
3. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree. (Hint: Use the pigeonhole principle.)
Let $G=(V, E)$ be a simple graph and $|V|=n$. Since G is a simple graph, it has neither cycles nor multi-edges. Hence, the degree of a vertex v in G must be between 0 and $n-1$, i.e., $0\le\deg(v)\le n-1$. However, if there exists a vertex with degree zero, all other vertices cannot have degree $n-1$. Namely, vertices of G cannot have degree zero and degree $n-1$ at the same time. Therefore, there are only $n-1$ choices of degrees for n vertices of G. By the pigeonhole principle, there must be at two vertices in V with the same degree.

4. Determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

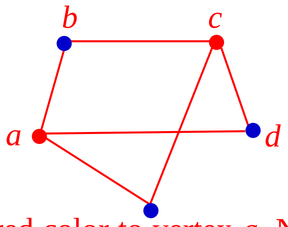
(a)

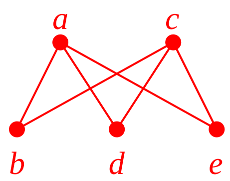


(b)

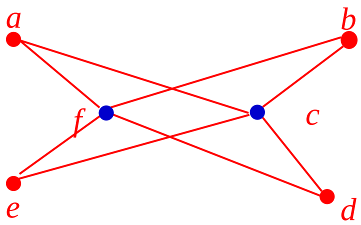


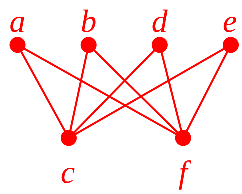
(a) First, assign red color to vertex *a*. Next, assign blue color to vertices *b*, *c*, and *d* which are adjacent to vertex *a*. Finally, assign read color to vertex *c*. As the graph shown on the left-hand-side, there are no two adjacent vertices having the same color. Hence, by Theorem 4, this graph is a bipartite. In fact, it is the complete bipartite $K_{2,3}$ as the graph shown on the right-hand-side.





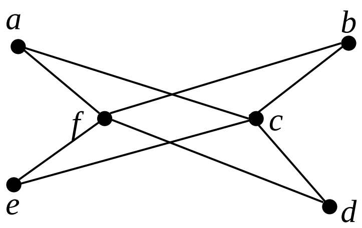
(b) First, assign red color to vertex *a*. Next, assign blue color to vertices *c*, and *f* which are adjacent to vertex *a*. Finally, assign read color to vertices *b*, *d*, and *e*. As the graph shown on the left-hand-side, there are no two adjacent vertices having the same color. Hence, by Theorem 4, this graph is a bipartite. In fact, it is the complete bipartite $K_{4,2}$ as the graph shown on the right-hand-side.



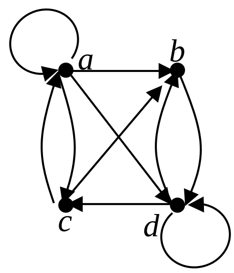


5. (i) Use an adjacency list to represent the following graphs. (ii) Use an adjacency matrix to represent the following graphs.

(a)



(b)



(i) Adjacency tables:

(a)

Vertex	Terminal Vertices
a	*c*, *f*
b	*c*, *f*
c	*a*, *b*, *d*, *e*
d	*c*, *f*
e	*c*, *f*
f	*a*, *b*, *d*, *e*

(b)

Vertex	Terminal Vertices
a	*a*, *b*, *c*, *d*
b	*d*
c	*a*, *b*
d	*b*, *c*, *d*

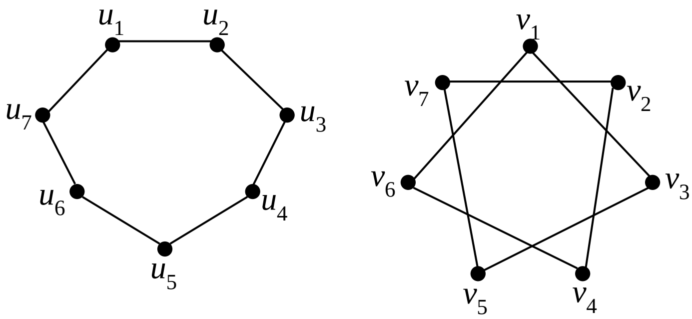
(ii) Adjacency matrixes:

(a) $$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

In Questions 6 to 8, if the pair of graphs is isomorphic, show the isomorphism function and the corresponding adjacency matrices; otherwise, provide a rigorous argument to explain why the pair of graphs is not isomorphic.

6. Determine whether the given pair of graphs is isomorphic.

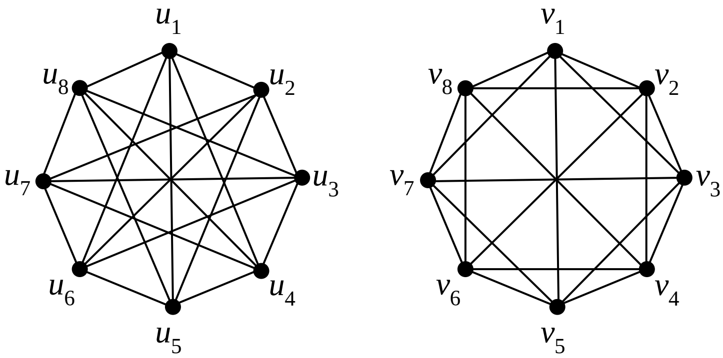


Both graphs have seven vertices and seven edges and each vertex is of degree two. We will try to find the isomorphism *f* of the two graphs. First, let us map vertex u_1 to v_1, $f(u_1)=v_1$. Then, we map their adjacent vertices, say, $f(u_2)=v_6$ and $f(u_7)=v_3$. Continually, we follow the adjacent vertices and obtain $f(u_3)=v_4$, $f(u_4)=v_2$, $f(u_5)=v_7$, $f(u_6)=v_5$. We already have $f(u_7)=v_3$. Hence, the two graphs *may be* isomorphic. To show they are indeed isomorphic, we will construct their corresponding adjacency matrices as the below:

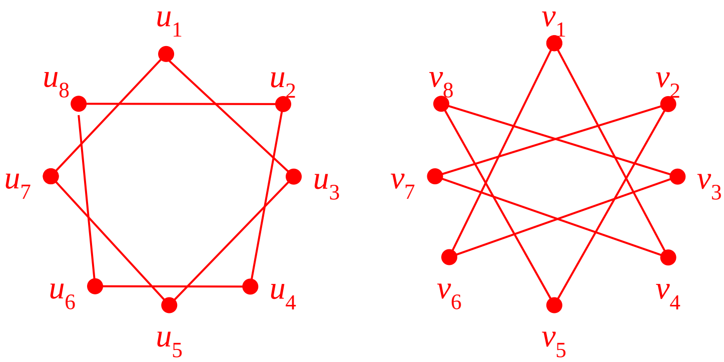
$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \text{, and } \begin{array}{c} \\ v_1 \\ v_6 \\ v_4 \\ v_2 \\ v_7 \\ v_5 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_6 & v_4 & v_2 & v_7 & v_5 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Since the two adjacency matrices are identical, it implies the two graphs are isomorphic.

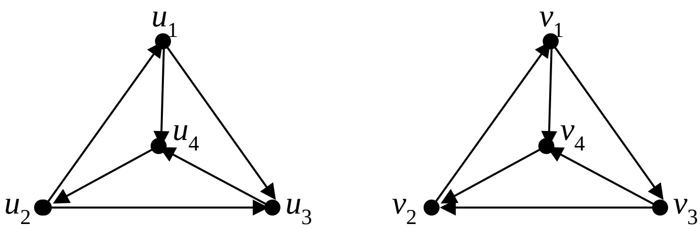
7. Determine whether the given pair of graphs is isomorphic.



The complement of graph $G = (V, E)$ is the graph $\bar{G} = (V, V \times V - E)$. Also we have the theorem that two graphs G and H are isomorphic if and only if their complements $\bar{G}$ and $\bar{H}$ are isomorphic. Consider the complements of the above graphs. Since the first complement has two cycles of length 4 and the second complement has one cycle of length 8, they are not isomorphic. Hence, the graphs given in the problem are not isomorphic.



8. Determine whether the given pair of directed graphs is isomorphic.



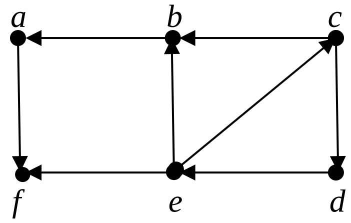
Both directed has four vertices and six edges. Each graph has two vertices with in-degree 2 and out-degree 1 and two vertices with in-degree 1 and out-degree 2. We will try to find the isomorphism *f* of the two graphs. First, let us map vertex u_1 to vertex v_1, $f(u_1)=v_1$, because both of them have in-degree 1 and out-degree 2. Since in the first graph, the incoming edge of vertex u_1 comes from vertex u_2, and in the second graph, the incoming edge of vertex v_1 comes from vertex v_2, we must map vertex u_2 to vertex v_2, $f(u_2)=v_2$. However, vertex u_2 is of in-degree 1 and out-degree 2 and vertex v_2 is of in-degree 2 and out-degree 1. This trial mapping is not an isomorphism.

We will then try to map vertex u_1 to another vertex of in-degree 1 and out-degree 2, i.e., vertex v_3, $f(u_1)=v_3$. In the result, the isomorphism is $f(u_2)=v_1$, $f(u_4)=v_2$, and $f(u_3)= v_4$. We construct the following adjacency matrices:

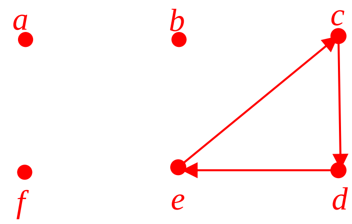
$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \text{ and } \begin{array}{c} \\ v_3 \\ v_1 \\ v_4 \\ v_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_3 & v_1 & v_4 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Since the two adjacency matrices are identical, the given graphs are isomorphic.

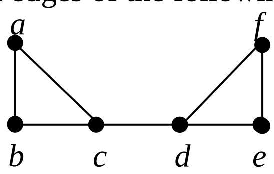
9. Find the strongly connected components of the following graph.



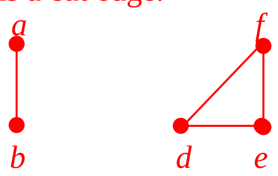
Since path *c*, *d*, *e*, *c* is a circuit, the circuit forms a strongly connected components. Since no pairs of vertices *a*, *b*, and *f* have paths connecting them in both directions, each of them forms a single-vertex strongly connected component. Hence, the strongly connected components are {*a*}, {*b*}, {*f*}, and {*c*, *d*, *e*} as below.



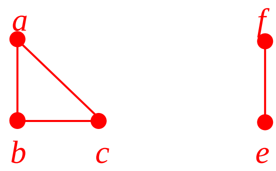
10. Find all the cut vertices and cut edges of the following graph.



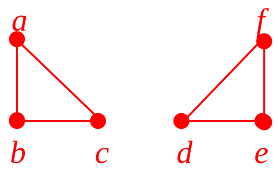
Removing either vertex *c* or vertex *d*, the graph will leave two components. Hence, either vertex *c* or vertex *d* is a cut vertex.
Removing edge {c, d}, the graph will leave two components. Hence, edge{*c*, *d*} is a cut edge.



Remove vertex *c* from the graph.

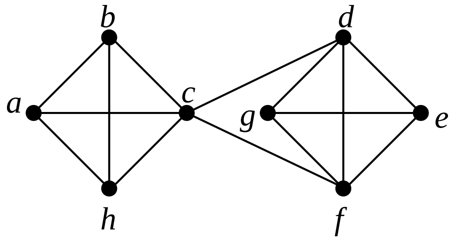


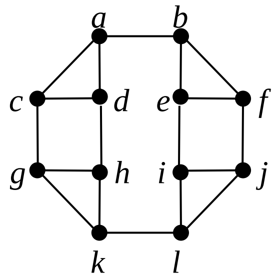
Remove vertex *d* from the graph.



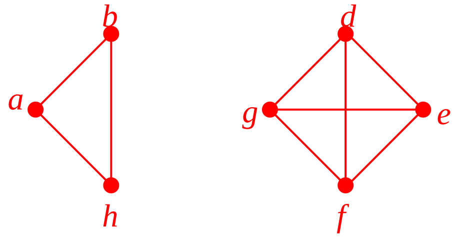
Remove edge {*c*, *d*} from the graph.

11. For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v\in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \le \lambda(G) \le \min_{v\in V} \deg(v)$ are strict.

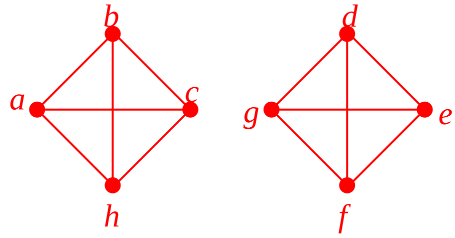




(a) Removing vertex *c*, the graph leaves two components. Hence, $\kappa(G)=1$. Removing one edge does not disconnect the graph. But, removing two edges $\{c, d\}$ and $\{c, f\}$, the graph becomes two components. Hence, $\lambda(G)=2$. The minimum degree of the graph is 3. Therefore, the inequality $\kappa(G)<\lambda(G)<\min_{v\in V} \deg(v)$ is strict.

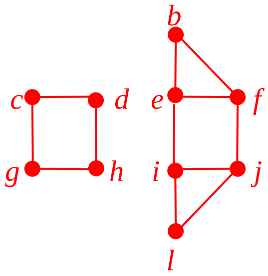


Remove vertex *c* from the graph.

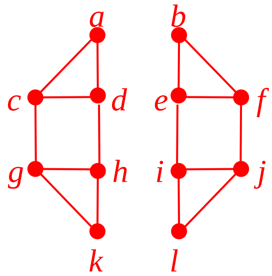


Remove edges $\{c, d\}$ and $\{c, f\}$ from the graph.

(b) Remove one vertex or one edge does not the graph. However, removing vertices *a* and *k* or removing edges $\{a, b\}$ and $\{k, l\}$, the graph becomes two disconnected components. Hence, $\kappa(G)=\lambda(G)=2$. The minimum degree of the graph is 3. Therefore, the inequality $\kappa(G)=\lambda(G)<\min_{v\in V} \deg(v)$ is not strict.



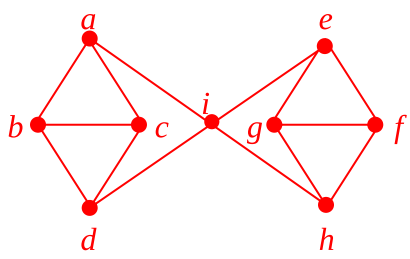
Remove vertices *a* and *k* from the graph.



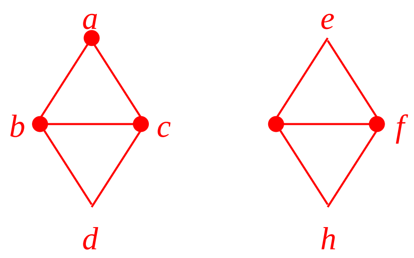
Remove edges $\{a, b\}$ and $\{k, l\}$ from the graph.

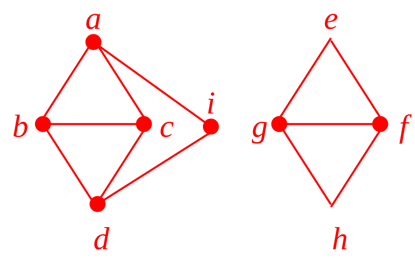
12. Construct a graph G with $\kappa(G)=1$, $\lambda(G)=2$, and $\min_{v\in V} \deg(v)=3$. Verify your solution.

The following graph is $\kappa(G)=1$, $\lambda(G)=2$, and $\min_{v\in V} \deg(v)=3$.



To verify the solution, we show the following graph with removal of vertex i and removal of edges $\{i, e\}$ and $\{i, h\}$.





Remove vertex i, so $\kappa=1$

Remove edges $\{i, e\}$ and $\{k, l\}$ from the graph, so $\lambda=2$.